LAGRANGE FORMALISM IN HEAT-CONDUCTION THEORY

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Hamilton's variational principle is formulated for an unsteady heat-conduction process in a solid incompressible medium. We establish the form of the Lagrange and Hamilton functions as well as of the canonic equations for the process described by a hyperbolic-type differential equation.

In recent years interest has grown in describing a heat-conduction process by the hyperbolic-type equation [1-3]

$$\frac{t_r}{a}\frac{\partial^2 T}{\partial t^2}+\frac{1}{a}\frac{\partial T}{\partial t}=\operatorname{div}\operatorname{grad} T, \tag{1}$$

that satisfies the finite heat-propagation velocity ($\omega_T = \sqrt{a/t_r}$) in a solid incompressible medium. According to Lykov [2, 3], the effect of the finite heat-propagation velocity is noticeable for gases under conditions of a rarefied supersonic flow. Solution (1) was successfully used in [4] to describe the heat-conduction process of dispersed systems.

We must note that even in the simplest cases (invariable thermophysical properties of the material, linear boundary conditions) the analytic solution of Eq. (1) is distinguished by its complexity [5]. Here we are interested in the possibility of applying a mathematical apparatus widely used in analytic mechanics — the so-called Lagrange formalism — to solve heat-conduction problems that can be determined by Eq. (1). Attempts to construct a Lagrange formalism to describe heat-conduction phenomena have been undertaken repeatedly.

Such an attempt is made in [6] with the equation

$$\frac{1}{a}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}, \tag{2}$$

that satisfies the heat propagation with an infinitely large velocity ($t_r \to 0$, $\omega_T \to \infty$). Here Morse and Feshbach proceed from the study of a pair of differential equations, one of which is the initial equation (2) and the other is conjugate to it.

Our method of introducing the conjugate equation is extended to the hyperbolic-type equation of heat conduction (1), and an analogy of phenomena in mechanics and heat conduction is developed on the basis of Hamilton's variational principle.

For definiteness we study a temperature field measured along one axis (the x axis) that is characterized by the equation

$$\frac{t_r}{a}\frac{\partial^2 T}{\partial t^2}+\frac{1}{a}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}. \tag{3}$$

We introduce an additional equation conjugate to (3),

$$\frac{t_r}{a}\frac{\partial^2 T_1}{\partial t^2}-\frac{1}{a}\frac{\partial T_1}{\partial t}=\frac{\partial^2 T_1}{\partial x^2}, \tag{4}$$

and write the system of equations (3) and (4) in dimensionless form,

$$\frac{\partial^2 q_1}{\partial X^2}-\frac{\partial q_1}{\partial \tau}-r\frac{\partial^2 q_1}{\partial \tau^2}=0, \tag{5}$$

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$$\frac{\partial^2 q_2}{\partial X^2}+\frac{\partial q_2}{\partial \tau}-r\frac{\partial^2 q_2}{\partial \tau^2}=0. \tag{6}$$

We introduce the Lagrangian function by the equation

$$\mathscr{L}=\mathscr{L}[q_1(X,\tau),\quad q_2(X,\tau)]=q_1' \, q_2'-r\dot{q}_1\dot{q}_2+\frac{1}{2}(q_2\dot{q}_1-q_1\dot{q}_2)-\varepsilon_0, \tag{7}$$

where

$$q_i'=\frac{\partial q_i}{\partial X};\quad \dot{q}_i=\frac{\partial q_i}{\partial \tau}\ (i=1,2)\ \text{and}\ \varepsilon_0=\text{const}.$$

We are easily assured that the assumed Lagrangian equation allows us to obtain the system of equations (5) and (6) by substituting (7) into Ostrogradskii's equations

$$\begin{aligned}&\frac{\partial \mathscr{L}}{\partial q_1}-\frac{\partial}{\partial \tau}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_1}\right)-\frac{\partial}{\partial X}\left(\frac{\partial \mathscr{L}}{\partial q_1'}\right)=0,\\ &\frac{\partial \mathscr{L}}{\partial q_2}-\frac{\partial}{\partial \tau}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_2}\right)-\frac{\partial}{\partial X}\left(\frac{\partial \mathscr{L}}{\partial q_2'}\right)=0.\end{aligned} \tag{8}$$

We recall that Hamilton's variational principle is formulated in analytic mechanics to describe the motion of the system of material points

$$\delta W=\delta\int_{t_1}^{t_2}L(t,\ q_i,\ \dot{q}_i)\,dt=0\quad (i=1,\ 2,\ \ldots,\ k), \tag{9}$$

where W is the Hamilton action; L is a Lagrange function; q_i and $\dot{q}_i = dq_i/dt$ are generalized coordinates and generalized velocities of points of the system; δ is a symbol of isochronic variation [7]. Using the variation procedure with the initial independence of the generalized coordinates taken into account, we can obtain from condition (9) a system of k differential equations of motion for the system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=0\quad (i=1,\ 2,\ \cdots,\ k), \tag{10}$$

which are called Lagrange equations of the second kind.

By introducing the momenta $p_i = \partial L/\partial \dot{q}_i$ and the Hamilton function $H=\sum_{i=1}^{k}p_i\dot{q}_i-L$, we can obtain equations of motion for the system of material points in another form,

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i},\quad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}. \tag{11}$$

Equations (11) are canonic Hamilton equations; here, for conservative systems, the Hamilton function H indicates the total energy of the system and remains invariable during its motion.

We formulate the Hamilton variational principle for the phenomena of unsteady heat conduction in a solid medium, as follows:

$$\delta W=\delta\int_{\tau_1}^{\tau_2}\int_{(V_0)}\mathscr{L}[q_1(X,\ \tau),\ q_2(X,\ \tau)]\ dV\,d\tau=0, \tag{12}$$

where V_0 = const is the volume of the body. In comparing Eqs. (9) and (12), we introduce the Lagrange function to describe the heat-conduction phenomena

$$L=\int_{(V_0)}\mathscr{L}[q_1(X,\ \tau),\ q_2(x,\ \tau)]\,dV. \tag{13}$$

We further develop the analogy of mechanics and thermodynamics and introduce the "momenta" by the equations

$$p_{1\tau} = \frac{\partial \mathscr{L}}{\partial \dot{q}_1}, \qquad p_{2\tau} = \frac{\partial \mathscr{L}}{\partial \dot{q}_2},$$
$$p_{1x} = \frac{\partial \mathscr{L}}{\partial q_1'}, \qquad p_{2x} = \frac{\partial \mathscr{L}}{\partial q_2'}. \tag{14}$$

Taking Eq. (7) into account, we obtain

$$p_{1\tau} = \frac{1}{2} q_2 - r\dot{q}_2, \quad p_{2\tau} = -\frac{1}{2} q_1 - r\dot{q}_1,$$
$$p_{1x} = q_2', \qquad p_{2x} = q_1'. \tag{15}$$

In addition, we introduce the Hamilton function

$$h = p_{1\tau}\dot{q}_1 + p_{2\tau}\dot{q}_2 + p_{1x}q_1' + p_{2x}q_2' - \mathscr{L}. \tag{16}$$

Taking Eqs. (7) and (15) into account, we find

$$h = q_1' q_2' - r\dot{q}_1\dot{q}_2 + \varepsilon_0. \tag{17}$$

We can show that the Hamilton function

$$H = V_0 (\varepsilon_0 + \int_0^1 q_1' q_2' dX - r\int_0^1 \dot{q}_1\dot{q}_2 dX) \tag{18}$$

remains invariable in the process and indicates the total energy of the system that can be determined by Eqs. (5) and (6).

In conclusion, we obtain a system of differential equations that describe the process of unsteady heat conduction and the canonic Hamilton equations (11) of analytic mechanics that appear by analogy. Here we proceed from the Hamilton principle (12). Taking Eq. (16) into account, we write the following for a body measuring along one axis (the x axis):

$$\delta \int_{\tau_1}^{\tau_2}\int_0^1 (p_{1\tau}\dot{q}_1 + p_{2\tau}\dot{q}_2 + p_{1X}q_1' + p_{2X}q_2' - h)\, dX d\tau = 0. \tag{19}$$

After performing the variation operations and taking into account that

$$\delta p_{1x} = \frac{dp_{1X}}{dq_2'}\, \delta q_2', \ \delta p_{2X} = \frac{dp_{2X}}{dq_1'}\, \delta q_1',$$
$$\delta p_{1\tau} = \frac{\partial p_{1\tau}}{\partial q_2}\, \delta q_2 + \frac{\partial p_{1\tau}}{\partial \dot{q}_2}\, \delta \dot{q}_2,$$
$$\delta p_{2\tau} = \frac{\partial p_{2\tau}}{\partial q_1}\, \delta q_1 + \frac{\partial p_{2\tau}}{\partial \dot{q}_1}\, \delta \dot{q}_1,$$

we find from Eq. (19)

$$0 = \int_{\tau_1}^{\tau}\int_0^1 \Big\{ \delta q_1 \Big[\frac{\partial p_{2\tau}}{\partial q_1}\dot{q}_2 - \frac{\partial}{\partial \tau}\Big(p_{1\tau} + \frac{\partial p_{2\tau}}{\partial \dot{q}_1}\dot{q}_2 - \frac{\partial h}{\partial \dot{q}_1}\Big) -$$
$$- \frac{\partial}{\partial X}\Big(p_{1X} + \frac{dp_{2X}}{dq_1'}\, q_2' - \frac{\partial h}{\partial q_1'}\Big)\Big] + \delta q_2 \Big[\frac{\partial p_{1\tau}}{\partial q_2}\dot{q}_1 -$$
$$- \frac{\partial}{\partial \tau}\Big(p_{2\tau} + \frac{\partial p_{1\tau}}{\partial q_2}\, \dot{q}_1 - \frac{\partial h}{\partial \dot{q}_2}\Big) - \frac{\partial}{\partial X}\Big(p_{2X} + \frac{dp_{1X}}{dq_2'}\, q_1' - \frac{\partial h}{\partial q_2'}\Big)\Big]\Big\} dX d\tau. \tag{20}$$

Due to the material independence of the "generalized coordinates" q_1 and q_2, the equations in brackets should become zero, and from this follows the desired system of "canonic equations,"

$$\frac{\partial p_{1\tau}}{\partial q_2}\dot{q}_1-\frac{\partial}{\partial\tau}\left(p_{2\tau}+\frac{\partial p_{1\tau}}{\partial \dot{q}_2}\dot{q}_1-\frac{\partial h}{\partial \dot{q}_2}\right)=\frac{\partial}{\partial X}\left(p_{2X}+\frac{dp_{1X}}{dq_2'}q_1'-\frac{\partial h}{\partial q_2'}\right), \tag{21}$$

$$\frac{\partial p_{2\tau}}{\partial q_1}\dot{q}_2-\frac{\partial}{\partial\tau}\left(p_{1\tau}+\frac{\partial p_{2\tau}}{\partial \dot{q}_1}\dot{q}_2-\frac{\partial h}{\partial \dot{q}_1}\right)=\frac{\partial}{\partial X}\left(p_{1X}+\frac{dp_{2X}}{dq_1'}q_2'-\frac{dh}{dq_1'}\right). \tag{22}$$

We see that the "canonic equations" (21) and (22) have a more complex form than the classical equations (11) of analytic mechanics. We are easily assured that in substituting the equations for the "momenta" from (15) and the Hamilton function from (17) into (21) and (22) we arrive at the initial system of equations (5) and (6). In this sense the "canonic equations" (21) and (22) are equivalent to the Ostrogradskii equations (8).

NOTATION

T, temperature; T_0 = const, initial temperature of a body; T_c = const, ambient temperature; ρ, c_V, λ; coefficients of mass density, specific heat capacity at a constant volume, and thermal conductivity of material; x, coordinate; l, linear characteristic dimension of a body along the x axis; t, time; t_r, relaxation time; ω_T, heat-propagation velocity; $X = x/l$, $\tau = (\lambda/\rho c_V)(t/l^2)$, $r = (\lambda/\rho c_V)(t_r/l^2)$, $q_1 = (T - T_c)/(T_0 - T_c)$, $q_2 = (T_1 - T_c)/(T_0 - T_c)$.

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